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THE D_1-TRIANGULATION IN SIMPLICIAL VARIABLE DIMENSION ALGORITHMS ON THE UNIT SIMPLEX FOR COMPUTING FIXED POINTS

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The D_1-Triangulation in Simplicial Variable Dimension Algorithms on the Unit Simplex for Computing Fixed Points

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Abstract

We consider how to use the D_1-triangulation in simplicial variable dimension algorithms on the unit simplex for computing fixed points. A new version of the D_1-triangulation is proposed. We can use directly this version in simplicial variable dimension algorithms on the unit simplex. According to measures of efficiency of triangulations, the D_1-triangulation is superior to the other well-known triangulations of R^n. So it is hoped that the cost of computation can be reduced through using the D_1-triangulation in simplicial variable dimension algorithms on the unit simplex.

Keywords: Simplicial Variable Dimension Algorithms, Triangulations, Measures of Efficiency of Triangulations

1 Introduction

In order to find out fixed points of nonlinear equations, a number of simplicial algorithms have been introduced. A simplicial algorithm subdivides the underlying space into simplices and searches for a simplex that yields an approximate solution. To find such a simplex, the algorithm generates a sequence of adjacent simplices. A simplicial variable dimension algorithm generates a sequence of simplices of varying dimension. This sequence connects an arbitrary starting point with an approximate solution. Since van der Laan and Talman proposed the first simplicial variable dimension algorithm without an extra dimension on the unit simplex $S^n = \left\{x \in R_+^{n+1} \mid \sum_{j=0}^{n} x_j = 1\right\}$ in [8], several contributions have been made, for example, their $(n+1)$-ray and $2n$-ray methods on R^n in [9], Wright's 2^n-ray method on R^n in [13], Kojima and Yamamoto's $(3^n - 1)$-ray method on R^n in [7], Yamamoto's 2-ray method on R^n in [14], Doup, van der Laan and Talman's $(2^{n+1} - 2)$-ray method on S^n in [5], and so on. For a survey on simplicial algorithms, see [1]. In [10], a simplicial variable dimension algorithm was introduced by Talman and Yamamoto to find a stationary point of a continuous function on a polytope. Doup gave an excellent survey about simplicial variable dimension algorithms on the product space of unit simplices in [4]. Recently, a new triangulation of R^n was proposed in [2], the D_1-triangulation. According to measures of efficiency of triangulations, it is superior to the other well-known triangulations. But it is not straightforward to use the D_1-triangulation in simplicial variable dimension algorithms except the $2n$-ray and 2-ray mehtods on R^n. In [3], we introduced a version of the D_1-triangulation for simplicial variable dimension algorithms on the unit simplex. But it has as shortcoming not to be able to induce a simplicial subdivision according to the D_1-triangulation of all the subsets into which a simplicial variable dimension algorithm subdivides the unit simplex. To reduce the cost of computation, we reconsider how to apply the D_1-triangulation to simplicial variable dimension algorithms on the unit simplex. We propose a new version of the D_1-triangulation, called the D_{v1}-triangulation. This version subdivides each of the subsets according to the D_1-triangulation into which a simplicial algorithm subdivides the unit simplex.

The second section introduces the D_1-triangulation. The third section gives the pivot rules of the D_{v1}-triangulation. The fourth section presents how to apply the D_{v1}-triangulation to simplicial variable dimension algorithms on the unit simplex.

2 The D_{v1}-Triangulation

In this section we describe the D_{v1}-triangulation. This simplicial subdivision is based on the D_1-triangulation of R^n introduced in [2]. The D_{v1}-triangulation will be used as the underlying subdivision for simplicial variable dimension algorithms on the unit simplex. An n-dimensional simplex is the convex hull of $n+1$ affinely independent points called its vertices, and a simplicial subdivision of a compact, convex set is the set of a finite number of simplices such that the intersection of each pair of simplices is either empty or a common face and such that the union of all simplices is the set itself.

Let m be a positive integer and let N denote the index set $\{1, 2, \ldots, n\}$. Furthermore, let

$$C^n(m) = \{x \in R^n \mid m \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

The D_{v1}-triangulation subdivides simplicially the set $C^n(m)$. The simplices of this triangulation can be represented by some specific vector y, permutation π, sign vector s, and integer p. Let D denote the set

$$\{y \in C^n(m) \mid \text{all components of } y \text{ are even}\}.$$

Take any $y \in D$. Let $1 \leq h \leq n$ be such that for $j = 1, 2, \ldots, h$, it holds that $y_{m_j+1} = y_{m_j+k}$ for $k = 2, 3, \ldots, n_j$, and for $j = 1, 2, \ldots, h-1$, it holds that $y_{m_j+n_j} > y_{m_{j+1}+1}$, where $m_1 = 0$, $n_h = n - m_h$, and $m_j + n_j = m_{j+1}$ for $j = 1, 2, \ldots, h-1$.

Take $s = (s_1, s_2, \ldots, s_n)^\top$ to be a sign vector such that

$$s_{m_j+1} = \cdots = s_{m_j+k_j} = 1$$

and

$$s_{m_j+k_j+1} = \cdots = s_{m_j+n_j} = -1$$

for $j = 1, 2, \ldots, h$, where $k_1 = 0$ if $y_1 = m$, $k_h = n_h$ if $y_n = 0$, and $0 \leq k_j \leq n_j$ for $j = 1, 2, \ldots, h$.

For $1 \leq j \leq h$ and $1 \leq k \leq k_j$, let n-vector $g(m_j + k)$ be given by

$$g_i(m_j + k) = \begin{cases} 1 & \text{if } m_j + 1 \leq i \leq m_j + k, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$.

For $1 \leq j \leq h$ and $k_j + 1 \leq k \leq n_j$, let n-vector $g(m_j + k)$ be given by

$$g_i(m_j + k) = \begin{cases} -1 & \text{if } m_j + k \leq i \leq m_j + n_j, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$.

Take an integer p such that $0 \leq p \leq n - 1$ and if $h = 1$ and $k_1 = 0$ or n then $p = 0$.

Finally, take a permutation π of the elements of N such that for $1 \leq j \leq h$,

$$\pi^{-1}(m_j + 1) > \pi^{-1}(m_j + 2) > \cdots > \pi^{-1}(m_j + k_j)$$

and

$$\pi^{-1}(m_j + k_j + 1) < \pi^{-1}(m_j + k_j + 2) < \cdots < \pi^{-1}(m_j + n_j)$$

and when $p \geq 1$, there exists no $1 \leq j \leq h$ such that $m_j + 1 \leq \pi(k) \leq m_j + k_j$ for all $p \leq k \leq n$ or $m_j + k_j + 1 \leq \pi(k) \leq m_j + n_j$ for all $p \leq k \leq n$.

Let u^i denote the i-th unit vector in R^n for $i = 1, 2, \ldots, n$.

Definition 2.1. Let the vector y, the permutation π, the sign vector s, and the number p be taken as above. Then the vectors y^0, y^1, ..., y^n are given as follows.

If $p = 0$, then $y^0 = y$,
$y^k = y + g(\pi(k)), k = 1, 2, \ldots, n$.
If $p \geq 1$, then $y^0 = y + s$,
$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, k = 1, 2, \ldots, p - 1$,
$y^k = y + g(\pi(k)), k = p, p + 1, \ldots, n$.

Let $y^0, y^1, \ldots, y^n$ be obtained from the above definition. Then it is obvious that they are affinely independent. Thus their convex hull is a simplex with vertices $y^0, y^1, \ldots, y^n$. Let us denote this simplex by $D_{v1}(y, \pi, s, p)$. Let D_{v1} be the set of all such simplices $D_{v1}(y, \pi, s, p)$. We will show that D_{v1} is a simplicial subdivision of $C^n(m)$.

For given y and s as above, let α be defined by $\alpha = \sum_{j=1}^{h} \alpha_j$, where

$$\alpha_j = \begin{cases} 2 & \text{if } 0 < k_j < n_j, \\ 1 & \text{otherwise,} \end{cases}$$

for $j = 1, 2, \ldots, h$.

For given y, π, s, and p as above, let

$$r_j = \mid \{m_j + k \mid 1 \le k \le k_j\} \cap \{\pi(k) \mid 1 \le k \le p-1\} \mid$$

and

$$q_j = \mid \{m_j + k \mid k_j + 1 \le k \le n_j\} \cap \{\pi(k) \mid 1 \le k \le p-1\} \mid$$

for $j = 1, 2, \ldots, h$.

Lemma 2.2. The union of all $\sigma \in D_{v1}$ is equal to $C^n(m)$.

Proof. Clearly, every $\sigma \in D_{v1}$ is a subset of $C^n(m)$. Let $x \in C^n(m)$ be arbitrary. Then $x \in D_{v1}(y, \pi, s, p)$ with y, π, s, and p determined as follows.

Take the vector y equal to

$$y_i = \begin{cases} \lfloor x_i \rfloor + 1 & \text{if } \lfloor x_i \rfloor \text{ is odd and } \lfloor x_i \rfloor < m, \\ \lfloor x_i \rfloor - 1 & \text{if } \lfloor x_i \rfloor \text{ is odd and } \lfloor x_i \rfloor = m, \\ \lfloor x_i \rfloor & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$ and the sign vector s equal to

$$s_i = \begin{cases} -1 & \text{if } \lfloor x_i \rfloor \text{ is odd and } \lfloor x_i \rfloor < m \\ & \text{or } \lfloor x_i \rfloor \text{ is even and } \lfloor x_i \rfloor = m, \\ 1 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$. It is obvious that $y \in D$. Take the permutation π such that

$$s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}) \le s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) \le \cdots \le s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)})$$

and for $1 \le j \le h$,

$$\pi^{-1}(m_j + 1) > \pi^{-1}(m_j + 2) > \cdots > \pi^{-1}(m_j + k_j)$$

and

$$\pi^{-1}(m_j + k_j + 1) < \pi^{-1}(m_j + k_j + 2) < \cdots < \pi^{-1}(m_j + n_j).$$

Suppose $\alpha = 1$. Then take $p = 0$. Now, let $y^0 = y$,

$$y^k = y + g(\pi(k)), k = 1, 2, \ldots, n,$$

and if $s_1 = 1$, let

$$\begin{aligned}
\beta_0 &= 1 - s_1(x_1 - y_1), \\
\beta_1 &= s_n(x_n - y_n), \\
\beta_2 &= s_{n-1}(x_{n-1} - y_{n-1}) - s_n(x_n - y_n), \\
&\ldots \\
\beta_n &= s_1(x_1 - y_1) - s_2(x_2 - y_2),
\end{aligned}$$

and if $s_1 = -1$, let

$$\begin{aligned}
\beta_0 &= 1 - s_n(x_n - y_n), \\
\beta_1 &= s_1(x_1 - y_1), \\
\beta_2 &= s_2(x_2 - y_2) - s_1(x_1 - y_1), \\
&\ldots \\
\beta_n &= s_n(x_n - y_n) - s_{n-1}(x_{n-1} - y_{n-1}).
\end{aligned}$$

Obviously, $\sum_{k=0}^{n} \beta_k = 1, \beta_k \geq 0$ for all k, and

$$x = \sum_{k=0}^{n} \beta_k y^k.$$

Hence,

$$x \in D_{v1}(y, \pi, s, p).$$

When $\alpha = n$, the proof is the same as that of **Lemma 2.2** in [2]. Now suppose that $1 < \alpha < n$. For $1 \leq j \leq h$, let

$$\mu_j = \begin{cases}
s_{m_j+1}(x_{m_j+1} - y_{m_j+1}) + s_{m_j+n_j}(x_{m_j+n_j} - y_{m_j+n_j}) \\
\qquad \text{if } \alpha_j = 2, \\
s_{m_j+1}(x_{m_j+1} - y_{m_j+1}) \\
\qquad \text{if } \alpha_j = 1 \text{ and } s_{m_j+1} = 1, \\
s_{m_j+n_j}(x_{m_j+n_j} - y_{m_j+n_j}) \\
\qquad \text{if } \alpha_j = 1 \text{ and } s_{m_j+1} = -1.
\end{cases}$$

Let $\mu = \sum_{j=1}^{h} \mu_j$.

In case $\mu \leq 1$. Take $p = 0$, and let $y^0 = y$,

$$y^k = y + g(\pi(k)), k = 1, 2, \ldots, n.$$

For $1 \leq k \leq n$, let

$$\beta_k = \begin{cases} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) \\ \quad \text{if } \pi(k) = m_j + k_j \text{ and } k_j \geq 1 \text{ or } \pi(k) = \\ \quad m_j + k_j + 1 \text{ and } k_j \leq n_j - 1 \text{ for some } 1 \leq j \leq h, \\ s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - s_{\pi(k)+1}(x_{\pi(k)+1} - y_{\pi(k)+1}) \\ \quad \text{if } 1 \leq \pi(k) - m_j < k_j \text{ for some } 1 \leq j \leq h, \\ s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - s_{\pi(k)-1}(x_{\pi(k)-1} - y_{\pi(k)-1}) \\ \quad \text{if } k_j + 1 < \pi(k) - m_j \leq n_j \text{ for some } 1 \leq j \leq h, \end{cases}$$

and let $\beta_0 = 1 - \mu$. Then it is clear that $\beta_k \geq 0$ for all k, $\sum_{k=0}^{n} \beta_k = 1$, and

$$x = \sum_{k=0}^{n} \beta_k y^k.$$

Thus

$$x \in D_{v1}(y, \pi, s, p).$$

Now suppose $\mu > 1$. Let p_{max} denote the largest $1 \leq p \leq n - 1$ such that there exists no $1 \leq j \leq h$ such that $m_j + 1 \leq \pi(k) \leq m_j + k_j$ for all $p \leq k \leq n$ or $m_j + k_j + 1 \leq \pi(k) \leq m_j + n_j$ for all $p \leq k \leq n$. We show that there exists $1 \leq p \leq p_{max}$ such that the following system has a nonnegative solution,

$$\begin{array}{lcl} \beta_0 & = & s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\ \beta_1 & = & s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\ \ldots & & \\ \beta_{p-2} & = & s_{\pi(p-1)}(x_{\pi(p-1)} - y_{\pi(p-1)}) \\ & & \quad - s_{\pi(p-2)}(x_{\pi(p-2)} - y_{\pi(p-2)}), \\ \beta_{p-1} & = & -s_{\pi(p-1)}(x_{\pi(p-1)} - y_{\pi(p-1)}) \\ & & \quad + (\sum_{i=p}^{n} \lambda_{\pi(i)} - 1)/(c(p) - 1), \end{array}$$

$$\beta_k = \begin{cases} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - s_{\pi(k)+1}(x_{\pi(k)+1} - y_{\pi(k)+1}) \\ \qquad \text{if } 1 \le \pi(k) - m_j < k_j - r_j \text{ for some } 1 \le j \le h, \\ s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - (\sum_{i=p}^{n} \lambda_{\pi(i)} - 1)/(c(p) - 1) \\ \qquad \text{if } 1 \le \pi(k) - m_j = k_j - r_j \\ \qquad \text{or } k_j + q_j + 1 = \pi(k) - m_j \le n_j \text{ for some } 1 \le j \le h, \\ s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - s_{\pi(k)-1}(x_{\pi(k)-1} - y_{\pi(k)-1}) \\ \qquad \text{if } k_j + q_j + 1 < \pi(k) - m_j \le n_j \text{ for some } 1 \le j \le h, \end{cases}$$

for $k = p, p+1, \ldots, n$, where for $i = p, \ldots, n$,

$$\lambda_{\pi(i)} = \begin{cases} 0 & \text{if } 1 < \pi(i) - m_j \le k_j - r_j \\ & \text{or } k_j + q_j + 1 \le \pi(i) - m_j < n_j \\ & \text{for some } 1 \le j \le h, \\ s_{m_j+1}(x_{m_j+1} - y_{m_j+1}) & \text{if } 1 = \pi(i) - m_j \le k_j - r_j \\ & \text{for some } 1 \le j \le h, \\ s_{m_j+n_j}(x_{m_j+n_j} - y_{m_j+n_j}) & \text{if } k_j + q_j + 1 \le \pi(i) - m_j = n_j \\ & \text{for some } 1 \le j \le h, \end{cases}$$

and

$$c(p) = \sum_{j=1}^{h} c_j(p)$$

with

$$c_j(p) = \begin{cases} 0 & \text{if } r_j = k_j \text{ and } q_j = n_j - k_j, \\ 2 & \text{if } r_j < k_j \text{ and } q_j < n_j - k_j, \\ 1 & \text{otherwise,} \end{cases}$$

for $j = 1, 2, \ldots, h$.

If $\beta_{p-1} \ge 0$ for $p = p_{max}$, it is clear that $\beta_k \ge 0$ for all k. If not, then since $\mu > 1$, there exists $1 \le p_0 \le p_{max} - 1$ such that

$$\begin{aligned} 0 \le \; & -s_{\pi(p_0-1)}(x_{\pi(p_0-1)} - y_{\pi(p_0-1)}) \\ & +(\textstyle\sum_{i=p_0}^{n} \lambda_{\pi(i)} - 1)/(c(p_0) - 1) \end{aligned}$$

and

$$\begin{aligned} 0 > \; & -s_{\pi(p_0)}(x_{\pi(p_0)} - y_{\pi(p_0)}) \\ & +(\textstyle\sum_{i=p_0+1}^{n} \lambda_{\pi(i)} - 1)/(c(p_0+1) - 1). \end{aligned}$$

Hence,

$$s_{\pi(p_0)}(x_{\pi(p_0)} - y_{\pi(p_0)}) - (\sum_{i=p_0}^{n} \lambda_{\pi(i)} - 1)/(c(p_0) - 1) \ge 0.$$

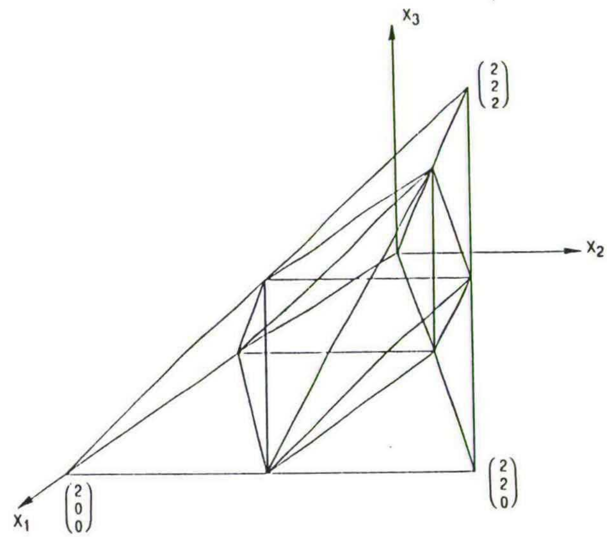
x_3
$\begin{pmatrix}2\\2\\2\end{pmatrix}$
x_2
x_1
$\begin{pmatrix}2\\0\\0\end{pmatrix}$
$\begin{pmatrix}2\\2\\0\end{pmatrix}$

FIGURE 1

Table 1: The Pivot Rules of the D_{v1}-Triangulation

i	p			$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$
0	0	$\alpha = 1, s_1 = 1$	$y_1 = m-1$	BD(1)			
			otherwise	$y + 2s_1 u^1$	$s - 2s_1 u^1$	π	$n-1$
		$\alpha = 1, s_1 = -1$		$y + 2s_n u^n$	$s - 2s_n u^n$	π	$n-1$
		$\alpha \geq 2$		y	s	π	$p+1$
0	1			y	s	π	$p-1$
0	$p \geq 2$		$y_{\pi(1)} = 0$ or m	BD(2)			
			otherwise	y	$s - 2s_{\pi(1)} u^{\pi(1)}$	π	p
$1 \leq i \leq n$	0	$1 \leq \pi(i) - m_j = k_j$ or $k_j + 1 = \pi(i) - m_j \leq n_j$ for some $1 \leq j \leq h$	$y_{\pi(i)} = 0$ or m	BD(3)			
			otherwise	y	$s - 2s_{\pi(i)} u^{\pi(i)}$	π	p
		otherwise		BD(4)			
$1 \leq i < p-1$		for some $1 \leq j \leq h$, $1 \leq \pi(k) - m_j \leq k_j$ for $k = i, i+1$ or $k_j + 1 \leq \pi(k) - m_j \leq n_j$ for $k = i, i+1$		BD(5)			
		otherwise		y	s	$(\pi(1), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$	p
$i = p-1$	$p \geq 2$			y	s	π	$p-1$
$p \leq i \leq n$	$1 \leq p < n-1$	for some $1 \leq j \leq h$, $1 \leq \pi(k) - m_j \leq k_j$ for $p \leq k \leq n$ and $k \neq i$ or $k_j + 1 \leq \pi(k) - m_j \leq n_j$ for $p \leq k \leq n$ and $k \neq i$	$i < n$ and either $y_{\pi(n)} \neq m-1$ or $s_{\pi(n)} \neq 1$	$y + 2s_{\pi(n)} u^{\pi(n)}$	$s - 2s_{\pi(n)} u^{\pi(n)}$	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$n-1$
			$i = n$ and either $y_{\pi(n-1)} \neq m-1$ or $s_{\pi(n-1)} \neq 1$	$y + 2s_{\pi(n-1)} u^{\pi(n-1)}$	$s - 2s_{\pi(n-1)} u^{\pi(n-1)}$	$(\pi(1), \ldots, \pi(p-1), \pi(n), \pi(p), \ldots, \pi(n-1))$	$n-1$
			otherwise	BD(6)			
		for some $1 \leq j \leq h$, $1 \leq \pi(i) - m_j = k_j - r_j$ or $k_j + q_j + 1 = \pi(i) - m_j \leq n_j$		y	s	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$p+1$
		otherwise		BD(7)			
	$1 \leq p = n-1$		$i = n-1$ and either $y_{\pi(n)} \neq m-1$ or $s_{\pi(n)} \neq 1$	$y + 2s_{\pi(n)} u^{\pi(n)}$	$s - 2s_{\pi(n)} u^{\pi(n)}$	π	p^*
			$i = n$ and either $y_{\pi(n-1)} \neq m-1$ or $s_{\pi(n-1)} \neq 1$	$y + 2s_{\pi(n-1)} u^{\pi(n-1)}$	$s - 2s_{\pi(n-1)} u^{\pi(n-1)}$	π^*	p^*
			otherwise	BD(8)			

Thus when $p = p_0$, $\beta_k \geq 0$ for all k. Obviously, $\sum_{k=0}^{n} \beta_k = 1$. Let $y^0 = y + s$, and for $k = 1, 2, \ldots, p-1$, let

$$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)},$$

and for $k = p, p+1, \ldots, n$, let

$$y^k = y + g(\pi(k)).$$

We easily obtain that $x = \sum_{k=0}^{n} \beta_k y^k$. Therefore,

$$x \in D_{v1}(y, \pi, s, p).$$

From the above conclusions, the lemma follows immediately.

Theorem 2.3. D_{v1} is a triangulation of $C^n(m)$.

Proof. From **Lemma 2.2** and the definition, the theorem follows immediately.

We call this simplicial subdivision of $C^n(m)$ the D_{v1}-triangulation. The D_{v1}-triangulation of $C^3(2)$ is illustrated in Figure 1. The D_{v1}-triangulation of $C^n(m)$ is such that the set $C^n(m)$ is subdivided into simplices according to the D_1-triangulation of R^n with grid size m^{-1} except the corners of the set. Moreover, every t-dimension face of $C^n(m)$ is also simplicially sudivided according to the D_1-triangulation of R^t, $1 \leq t \leq n-1$. Therefore, the D_{v1}-triangulation of $C^n(m)$ is superior to all other triangulations of this set, see [2].

3 The Pivot Rules of the D_{v1}-Triangulation

Let $\sigma = D_{v1}(y, \pi, s, p)$ be a simplex of the D_{v1}-triangulation of $C^n(m)$ with vertices y^0, y^1, $\ldots$, y^n. We want to obtain the parameters of the simplex $\bar{\sigma} = D_{v1}(\bar{y}, \bar{\pi}, \bar{s}, \bar{p})$ sharing with σ the facet opposite to the vertex y^i, $i \in \{0, 1, \ldots, n\}$, of σ, in case this facet of σ does not lie in the boundary of $C^n(m)$. In Table 1, we show how $\bar{y}, \bar{\pi}, \bar{s}$ and $\bar{p}$ depend on y, π, s, p and i. From this table, it is easy to obtain each vertex of $\bar{\sigma}$, and in particular the vertex opposite to the facet shared with σ.

In this table, π^* and p^* denote that if, for some $1 \leq j \leq \bar{h}$,

$$1 \leq \pi(k) - \bar{m}_j \leq \bar{k}_j \text{ for } k = n-1, n$$

or

$$\bar{k}_j + 1 \leq \pi(k) - \bar{m}_j \leq \bar{n}_j \text{ for } k = n-1, n,$$

then

$$\pi^* = (\pi(1), \ldots, \pi(n-2), \pi(n), \pi(n-1))$$

and

$$p^* = \begin{cases} k & \text{if there exists } 1 \leq k \leq n-2 \text{ such that } 1 \leq \pi(l) - \bar{m}_j \leq \bar{k}_j \\ & \text{for all } k < l \leq n \text{ and } \pi(k) \text{ doesn't or} \\ & \bar{k}_j + 1 \leq \pi(l) - \bar{m}_j \leq \bar{n}_j \\ & \text{for all } k < l \leq n \text{ and } \pi(k) \text{ doesn't,} \\ 0 & \text{otherwise;} \end{cases}$$

otherwise, $\pi^* = \pi$ and $p^* = p$, where $\bar{h}$, $\bar{m}_j$, $\bar{k}_j$, and $\bar{n}_j$ satisfy that for $j = 1, 2, \ldots, \bar{h}$, it holds that $\bar{y}_{\bar{m}_j+1} = \bar{y}_{\bar{m}_j+k}$ for $k = 2, 3, \ldots, \bar{n}_j$, and for $j = 1, 2, \ldots, \bar{h}-1$, it holds that $\bar{y}_{\bar{m}_j+\bar{n}_j} > \bar{y}_{\bar{m}_{j+1}+1}$, where $\bar{m}_1 = 0$, $\bar{n}_{\bar{h}} = n - \bar{m}_{\bar{h}}$, and $\bar{m}_j + \bar{n}_j = \bar{m}_{j+1}$ for $j = 1, 2, \ldots, \bar{h}-1$, and for $j = 1, 2, \ldots, \bar{h}$, it holds that $\bar{s}_{\bar{m}_j+1} = \cdots = \bar{s}_{\bar{m}_j+\bar{k}_j} = 1$ and $\bar{s}_{\bar{m}_j+\bar{k}_j+1} = \cdots = \bar{s}_{\bar{m}_j+\bar{n}_j} = -1$, where $\bar{k}_1 = 0$ if $\bar{y}_1 = m$, and $\bar{k}_{\bar{h}} = \bar{n}_{\bar{h}}$ if $\bar{y}_n = 0$.

4 Simplicial Variable Dimension Algorithms Based on the D_{v1}-Triangulation

Here we only consider how to use the D_{v1}-triangulation in the $(2^{n+1} - 2)$-ray method proposed by Doup, van der Laan and Talman in [5]. It can similarly be derived how to use the D_{v1}-triangulation in the other simplicial variable dimension algorithms on the unit simplex.

Let N_0 denote the index set $\{0, 1, \ldots, n\}$ and let S^n denote the unit simplex

$$\left\{ x \in R^{n+1} \mid \sum_{j=0}^{n} x_j = 1, x_j \geq 0 \text{ for all } j \in N_0 \right\}.$$

Let $f : S^n \to R^{n+1}$ be continuous on S^n such that $x^\top f(x) = 0$ for all $x \in S^n$, i.e., f is a complementary function on S^n. Our purpose is to find a point x^*

in S^n such that $f(x^*) \leq 0$. Take arbitrarily $x^0 \in S^n$ as an initial point. Let H be a nonempty subset of N_0 and let

$$S^n(H) = \{x \in S^n \mid x_j = 0 \text{ for all } j \notin H\}.$$

Set $H_0 = \left\{j \in H \mid x_j^0 = 0\right\}$. Then the projection vector of x^0 on $S^n(H)$, v(H), is defined by

$$v_i(H) = \begin{cases} 0 & \text{if } i \notin H, \\ (1 - \sum_{j\in H} x_j^0)/(\sum_{j\in H} x_j^0 + \mid H_0 \mid) & \text{if } i \in H_0, \\ x_i^0(1 + \mid H_0 \mid)/(\sum_{j\in H} x_j^0 + \mid H_0 \mid) & \text{otherwise,} \end{cases}$$

for $i = 0, 1, \ldots, n$. When H is empty, we define $v(H) = x^0$.

Let $g = (g_0, g_1, \ldots, g_n)^\top$ be a sign vector such that $g_i \in \{-1, 0, +1\}$ for $i = 0, 1, \ldots, n$. Let $I^-(g) = \{i \in N_0 \mid g_i = -1\}$, $I^0(g) = \{i \in N_0 \mid g_i = 0\}$ and $I^+(g) = \{i \in N_0 \mid g_i = +1\}$. For a sign vector g such that both $I^+(g)$ and $\{i \in I^-(g) \mid x_i^0 > 0\}$ are nonempty, we define

$$A(g) = \left\{ x \in S^n \;\middle|\; \begin{array}{l} x_k/x_k^0 = 1 + \alpha \text{ if } k \in I^+(g) \text{ and } x_k^0 > 0, \\ \beta \leq x_k/x_k^0 \leq 1 + \alpha \text{ if } k \in I^0(g) \text{ and } x_k^0 > 0, \\ \beta = x_k/x_k^0 \text{ if } k \in I^-(g) \text{ and } x_k^0 > 0, \\ x_k = \alpha \text{ if } k \in I^+(g) \text{ and } x_k^0 = 0, \\ 0 \leq x_k \leq \alpha \text{ if } k \in I^0(g) \text{ and } x_k^0 = 0, \\ 0 = x_k \text{ if } k \in I^-(g) \text{ and } x_k^0 = 0, \text{ and} \\ 0 \leq \beta \leq 1 \leq 1 + \alpha \end{array} \right\}.$$

It is obvious that $A(g)$ is a t-dimensional polyhedron with $t = |I^0(g)| + 1$ and that S^n is equal to the union of $A(g)$ over all sign vectors g such that $|I^0(g)| = n - 1$. In addition, let $k_0 = 0$ and let $\gamma(g) = (k_1, k_2, \ldots, k_{t-1})$ be a permutation of the $t - 1$ elements in $I^0(g)$. Then we define

$$A(g, \gamma(g)) = \left\{ x \in S^n \;\middle|\; \begin{array}{l} x = x^0 + \sum_{j=0}^{t-1} \lambda(k_j) w(k_j) \\ \text{with } 0 \leq \lambda(k_{t-1}) \leq \cdots \leq \lambda(k_0) \leq 1 \end{array} \right\},$$

where $w(k_0) = v(I^+(g) \cup \{k_1, \ldots, k_{t-1}\}) - x^0$ and

$$w(k_j) = v(I^+(g) \cup \{k_1, \ldots, k_{t-j-1}\}) - v(I^+(g) \cup \{k_1, \ldots, k_{t-j}\})$$

for $j = 1, 2, \ldots, t - 1$. Then it is clear that $A(g, \gamma(g))$ is t-dimensional and that $A(g)$ is the union of $A(g, \gamma(g))$ over all permutations $\gamma(g)$ of the elements in $I^0(g)$. Let P be the $(n+1)\times t$-matrix with the i-th column equal to $w(k_{i-1})$ for $i = 1, 2, \ldots, t$. Then

$$A(g, \gamma(g)) = \left\{x^0 + m^{-1}Px \mid x \in C^t(m)\right\}.$$

This means that $A(g, \gamma(g))$ is homeomorphic to $C^t(m)$. Thus the D_{v1}-triangulation induces a simplicial subdivision of $A(g, \gamma(g))$, which is denoted by $D_{v1}(g, \gamma(g))$. Moreover, as can easily be shown, the union of $D_{v1}(g, \gamma(g))$ over all permutations of the elements in $I^0(g)$ is a simplicial subdivision of $A(g)$. We write it as $D_{v1}(g)$. Finally, a simplicial subdivision of S^n has been obtained by taking the union of $D_{v1}(g)$ over all sign vectors g such that both $I^+(g)$ and $\{i \in I^-(g) \mid x_i^0 > 0\}$ are nonempty.

For $x \in R^{n+1}$, let the sign vector of x be defined by

$$\mathrm{sgn}(x) = (\mathrm{sgn}_0(x), \mathrm{sgn}_1(x), \ldots, \mathrm{sgn}_n(x))^{\mathsf{T}},$$

where

$$\mathrm{sgn}_i(x) = \begin{cases} -1 & \text{if } x_i < 0, \\ 0 & \text{if } x_i = 0, \\ +1 & \text{if } x_i = +1, \end{cases}$$

for $i = 0, 1, \ldots, n$. Let $e(j)$ denote the j-th unit vector in R^{n+1} for $j = 0, 1, \ldots, n$. For a given sign vector g with $t = |I^0(g)| + 1$, a k-simplex σ with vertices z^0, z^1, …, z^k, for $k = t-1$ and t, is called g-complete if the $(n+2)$-system of linear equations

$$\sum_{j=0}^{k} \lambda_j \begin{pmatrix} f(z^j) \\ 1 \end{pmatrix} - \sum_{j \notin I^0(g)} \mu_j g_j \begin{pmatrix} e(j) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

has a nonnegative solution.

Starting at x^0 with $g = \mathrm{sgn}(f(x^0))$, the $(2^{n+1}-2)$-ray method generates a sequence of adjacent g-complete t-simplices in $A(g)$ with g-complete common facets for varying sign vectors g. Below we describe the steps of the algorithm in case the simplicial subdivision of each set $A(g)$ is based on the D_{v1}-triangulation for some given grid size m^{-1}.

The $(2^{n+1}-2)$-Ray Method Based on the D_{v1}-Triangulation:

Initialization: Without loss of generality we assume that $f_i(x^0) \neq 0$ for all

i. Set $g = \text{sgn}(f(x^0))$ and $t = 1$. Set $y = 0$, $\pi = (1)$, $s = 1$ and $p = 0$. Further, set $z^0 = x^0$ and $\tau_0 = \{z^0\}$. Finally, set $k = 0$.

Setp 1: Let σ_k be the simplex in $D_{v1}(g)$ corresponding to the simplex

$$D_{v1}(y, \pi, s, p).$$

Thus τ_k is a facet of σ_k. Let z^+ denote the vertex of σ_k opposite to τ_k. Perform a linear programming step with $l^+ = (f(z^+), 1)^\top$ in the system of linear equations

$$\sum_{j=0}^{t-1} \lambda_j \begin{pmatrix} f(z^j) \\ 1 \end{pmatrix} - \sum_{j \notin I^0(g)} \mu_j g_j \begin{pmatrix} e(j) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If some μ_c becomes zero, then set $z^t = z^+$ and go to **Step 2**; otherwise, some λ_d becomes zero, then set $z^- = z^d$ and $z^d = z^+$, and go to **Step 3**.

Step 2: When $I^+(g) = \{c\}$ or $\left\{j \in I^-(g) \mid x_j^0 > 0\right\} = \{c\}$, σ_k yields an approximate solution and the algorithm terminates; otherwise perform the following increasing dimension procedure. Set $\tau_{k+1} = \sigma_k$ and $k = k + 1$ and set $g_c = 0$. When $c \in I^+(g)$, set $\gamma(g) = (c, k_1, \ldots, k_{t-1})$, $y_{t+1} = 0$ and $s_{t+1} = 1$ and if $p = 0$ then set $\pi = (t+1, \pi(1), \ldots, \pi(t))$ and if $p \geq 1$ then set $\pi = (t+1, \pi(1), \ldots, \pi(t))$ and $p = p + 1$. When $c \in I^-(g)$, set $\gamma(g) = (k_1, \ldots, k_{t-1}, c)$, $y_i = y_{i-1}$ and $s_i = s_{i-1}$ for $i = t+1, \ldots, 2$, $y_1 = y_2$ and $s_1 = s_2$, set $p = p+1$ if $\pi^{-1}(1) \leq p - 1$, set $\pi = (\pi(1)+1, \ldots, \pi(\pi^{-1}(1)-1)+1, 2, 1, \pi(\pi^{-1}(1)+1)+1, \ldots, \pi(t)+1)$ in case $s_1 = 1$, and set $\pi = (\pi(1)+1, \ldots, \pi(\pi^{-1}(1)-1)+1, 1, 2, \pi(\pi^{-1}(1)+1)+1, \ldots, \pi(t)+1)$ in case $s_1 = -1$. Set $t = t + 1$, and go to **Step 1**.

Step 3: Let y^i be the vertex of $D_{v1}(y, \pi, s, p)$ corresponding to the vertex z^-. Set τ_{k+1} equal to the facet of σ_k opposite to the vertex z^-. Consider **Table 1**. If one of the cases BD(j), $j = 1, 2, \ldots, 8$, occurs, then τ_{k+1} lies in the boundary of $A(g, \gamma(g))$. Set $k = k + 1$.

1. When one of the cases BD(1), BD(2) and $y_{\pi(1)} = m$, BD(3) and $y_{\pi(i)} = m$, BD(6), or BD(8) occurs, τ_k yields an approximate solution, and the algorithm terminates.

2. When the case both BD(2) and $y_{\pi(1)} = 0$ occurs, set $\kappa = k_1$ and set $g_{k_1} = +1$, $k_j = k_{j+1}$ for $j = 1, 2, \ldots, t-2$, $\pi = (\pi(2), \pi(3), \ldots, \pi(t))$, $p = p - 1$, and $t = t - 1$, and go to **Step 4**.
3. When the case both BD(3) and $y_{\pi(i)} = 0$ occurs, set $\kappa = k_1$ and set $g_{k_1} = +1$, $k_j = k_{j+1}$ for $j = 1, 2, \ldots, t-2$, $\pi = (\pi(2), \pi(3), \ldots, \pi(t))$, and $t = t - 1$, and go to **Step 4**.
4. When either the case both BD(4) and either $\pi(i) = 1$ and $s_1 = 1$ or $\pi(i) = 2$ and $s_2 = -1$ or the case both BD(7) and either $\pi(i) = 1$ and $s_1 = 1$ or $\pi(i) = 2$ and $s_2 = -1$ occurs, set $\kappa = k_{t-1}$ and set $g_{k_{t-1}} = -1$, $y_j = y_{j+1}$ and $s_j = s_{j+1}$ for $j = 2, \ldots, t-1$, $\pi(j) = \pi(j) - 1$ if $\pi(j) \neq 1$ and 2 for $j < \max\{\pi^{-1}(1), \pi^{-1}(2)\}$, $\pi(\min\{\pi^{-1}(1), \pi^{-1}(2)\}) = 1$ and $\pi(j-1) = \pi(j) - 1$ for $\max\{\pi^{-1}(1), \pi^{-1}(2)\} < j \leq t$, and $t = t - 1$, and go to **Step 4**.
5. When the case both BD(5) and that either $\pi(i)$ or $\pi(i+1)$ is equal to one occurs, set $\kappa = k_{t-1}$ and set $g_{k_{t-1}} = -1$, $y_j = y_{j+1}$ and $s_j = s_{j+1}$ for $j = 2, \ldots, t-1$, $\pi(j) = \pi(j) - 1$ if $\pi(j) \neq 1$ and 2 for $j < \max\{\pi^{-1}(1), \pi^{-1}(2)\}$, $\pi(\min\{\pi^{-1}(1), \pi^{-1}(2)\}) = 1$ and $\pi(j-1) = \pi(j) - 1$ for $\max\{\pi^{-1}(1), \pi^{-1}(2)\} < j \leq t$, $p = p - 1$, and $t = t - 1$, and go to **Step 4**.
6. When one of the cases 1) both BD(4) and neither $\pi(i) = 1$ and $s_1 = 1$ nor $\pi(i) = 2$ and $s_2 = -1$, 2) both BD(7) and neither $\pi(i) = 1$ and $s_1 = 1$ nor $\pi(i) = 2$ and $s_2 = -1$, or 3) both BD(5) and that both $\pi(i)$ and $\pi(i+1)$ are unequal to one occurs, set

   $$\gamma(g) = (k_1, \ldots, k_{t-\pi(i)}, k_{t-\pi(i)-1}, \ldots, k_{t-1})$$

   if $s_{\pi(i)} = +1$ and set

   $$\gamma(g) = (k_1, \ldots, k_{t-\pi(i)+1}, k_{t-\pi(i)}, \ldots, k_{t-1})$$

   if $s_{\pi(i)} = -1$, and go to **Step 1**.

When none of the cases BD(j), $j = 1, 2, \ldots, 8$, occurs, set $y = \bar{y}$, $\pi = \bar{\pi}$, $s = \bar{s}$, and $p = \bar{p}$ according to **Table 1**, and go to **Step 1**.

Step 4: Set σ_{k+1} equal to τ_k and $k = k + 1$, and perform a linear programming step with $-g_\kappa(e(\kappa), 0)^\top$ in the system of linear equations

$$\sum_{j=0}^{t} \lambda_j \begin{pmatrix} f(z^j) \\ 1 \end{pmatrix} - \sum_{j \notin I^0(g), j \neq \kappa} \mu_j g_j \begin{pmatrix} e(j) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If some μ_c becomes zero, then go to **Step 2**; otherwise, some λ_d becomes zero, then set $z^- = z^d$, and go to **Step 3**.

Assuming nondegeneracy in each linear programming step, the algorithm terminates within a finite number of iterations in **Step 2** or the case 1 of **Step 3** with some simplex σ_k. Let $z^0, z^1, \ldots, z^t$, be the vertices of σ_k, then $x^1 = \sum_{j=0}^{t} \lambda_j z^j$ lies in σ_k for $\lambda_j \geq 0$ for all j such that $\sum_{j=0}^{t} \lambda_j = 1$. Moreover, the vector x^1 can be considered as an approximate complementary point of f on S^n. If the accuracy of approximation is not satisfactory, then the $(2^{n+1} - 2)$-ray method can be restarted at x^1 for some larger m, in order to improve the accuracy, as has been shown in [5]. Since the D_1-triangulation is superior to all other well-known triangulations of R^n according to measures of efficiency of triangulations, e.g., see [11] and [12], it is hoped that when using the D_{v1}-triangulation in a simplicial variable dimension algorithm on the unit simplex the cost of computation can be reduced.

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